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FUZZY MODELS OF FIRST ORDER LANGUAGES

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1. Introduction

In order to give a general approach to fuzzy set theory, we utilize the concepts of generalized algebra and realization (in this paper we call them *valuation structure* and *fuzzy model*, respectively) given by H. RASIOWA and R. SIKORSKI in [13], [15] and others papers. Namely we treat the valuation structures of a given type and the fuzzy models of a given first order language in a suitable categorial setting and we prove that these categories have direct products. Also we define and examine congruence and quotient concepts for valuation structures and fuzzy models. Moreover, we prove that direct product and quotient operations preserve the first order properties (see also [16]). Still a very general concept of quantifier enable us to propose new definitions of fuzzy entropy (degree of fuzziness) for a fuzzy relation. Finally, by utilizing the quotient concept, we examine the question of associating to a fuzzy model $\mathcal{M}$ a suitable "crisp version", i.e. a classic model $\mathcal{M}'$ with the "same" first order properties of $\mathcal{M}$. In order to illustrate, by an example, the possible applications of the above results, we refer constantly ourselves to the fuzzy algebras [14], [4], even if all the results hold for many other objects of investigation of fuzzy set theory. For example they hold for the free, pure, left unitary, right unitary, unitary fuzzy subsemigroups [7], [8], [10], moreover for fuzzy graphs and similarity relations [11].

2. The category of the valuation structures

We denote by N the set of natural numbers and by N^* the set $\mathsf{N} \setminus \{0\}$. A type for a valuation structure is a family of disjoint sets $\tau = \langle \bar{Q}, \bar{C}_0, \bar{C}_1, \ldots, \bar{C}_n, \ldots \rangle$. If $\bar{Q} \neq \emptyset$, the elements of $\bar{Q}$ are called *quantifiers*, if $\bar{C}_n \neq \emptyset$, the elements of $\bar{C}_n$ are called *n-ary connectives*. A *valuation structure* or *generalized algebra* (see [13], [15]) of type τ is a pair $\mathcal{V} = (V, I)$, where V is a set (the true values set) and I (the interpretation) is a map defined in $\bar{Q} \cup \bigcup\{\bar{C}_n \mid n \in \mathsf{N}\}$ such that I associates

a) to every $\bar{c} \in \bar{C}_n$ an n-ary operation $c = I(\bar{c})$ of V,

b) to every quantifier $\bar{q} \in \bar{Q}$ a map $q = I(\bar{q})$ from a class D_q of subsets of V to V; D_q is called the *domain* of q.

We set $Q = \{I(\bar{q}) \mid \bar{q} \in \bar{Q}\}$, $C_n = \{I(\bar{c}) \mid \bar{c} \in \bar{C}_n\}$ and $C = \bigcup\{C_n \mid n \in \mathsf{N}\}$. We can also denote $\mathcal{V} = (V, I)$ by (V, C, Q). In other words a valuation structure is determined by an ordinary algebraic structure (V, C) and, if $\bar{Q} \neq \emptyset$, by a set of infinitary operations Q. A valuation structure is *complete* if $D_q = \mathcal{P}(V)$ for every $q \in Q$.

If (V, I) and (V', I') are two valuation structures of the same type, then a *homomorphism* from (V, I) to (V', I') is a map $k\colon V \to V'$ such that for every $\bar{c} \in \bar{C}_n$, $\bar{q} \in \bar{Q}$, $v_1, \ldots, v_n \in V$, $X \in D_q$

(2.1) $k(c(v_1, \ldots, v_n)) = c'(k(v_1), \ldots, k(v_n))$,

(2.2) $k(X) \in D_{q'}$ and $k(q(X)) = q'(k(X))$,

where $c = I(\bar{c})$, $c' = I'(\bar{c})$, $q = I(\bar{q})$, $q' = I'(\bar{q})$.

It is obvious that the valuation structures of a given type τ constitute a category $\boldsymbol{V}(\tau)$ with respect the above defined homomorphisms. In the case $\bar{Q} = \{\exists, \forall\}$, $\bar{C}_0 = \{t, f\}$, $\bar{C}_1 = \{\sim\}$ and $\bar{C}_2 = \{\vee, \wedge, \rightarrow\}$ there exist many interesting examples of full subcategories of $\boldsymbol{V}(\tau)$. For example the categories of the Boolean, the Łukasiewicz or the Heyting algebras, where $I(\forall)$ and $I(\exists)$ are infimum and supremum operators, respectively. If $\bar{C}_1 = \{\sim, \Box\}$, we can obtain the categories of the modal algebras. All these categories are a tool to found a semantic for the corresponding first order logic. In the sequel we denote by $\mathscr{B}_2$ the two elements Boolean algebra and by $\mathscr{B}_\infty$ the Łukasiewicz algebra (V_∞, I_∞), where V_∞ is the real unit interval $[0, 1]$ and $I_\infty(t) = 1$, $I_\infty(f) = 0$, $I_\infty(\wedge)(x, y) = \min(x, y)$, $I_\infty(\vee)(x, y) = \max(x, y)$, $I_\infty(\sim)(x) = 1 - x$, $I_\infty(\forall)(M) = \inf M$, $I_\infty(\exists)(M) = \sup M$, for $x, y \in [0, 1]$ and $M \subseteq [0, 1]$.

Let S be a set and $\mathscr{V}$ a valuation structure. An *n-ary fuzzy relation* or *$\mathscr{V}$-relation* is any map $r\colon S^n \to V$. An 1-ary fuzzy relation is also named *fuzzy subset* or *$\mathscr{V}$-subset* of S. If $\mathscr{V}$ is ordered and $u \in V$, then $C(u) = \{(x_1, \ldots, x_n) \in S^n \mid r(x_1, \ldots, x_n) \geqq u\}$ is called the *u-cut* of r. If $\mathscr{V}$ is an ordered set with minimum 0 and maximum 1, then a $\mathscr{V}$-relation r such that $r(x) \in \{0, 1\}$ for every $x \in S$ is called *crisp*. Then we can identify the classical relations with the crisp relations via the characteristic functions. All these definitions are obvious generalizations of those given in literature when $\mathscr{V}$ is the algebra $\mathscr{B}_\infty$ [17] or a lattice [11]. The following proposition shows that the category of the valuation structures of a given type has direct products.

Proposition 2.1. *The category $\boldsymbol{V}(\tau)$ of the valuation structures of a given type τ has direct products. Namely, let $(\mathscr{V}_i)_{i\in J}$ be a family of objects of $\boldsymbol{V}(\tau)$, where $\mathscr{V}_i = (V_i, I_i) = (V_i, C_i, Q_i)$, and let $\mathscr{V} = (V, I) = (V, C, Q)$ be defined by*

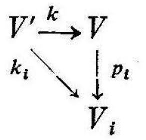
(i) *(V, C) equal to the direct product of $\langle(V_i, C_i)\rangle_{i\in J}$,*

(ii) *$D_q = \{X \subseteq \mathscr{V} \mid$ for every $i \in J$: $p_i(X) \in D_{q_i}\}$,*

(iii) *$q(X) = \langle q_i(p_i(X))\rangle_{i\in J}$,*

where $p_i\colon V \to V_i$ is the i-projection, $\bar{q} \in \bar{Q}$, $q_i = I_i(\bar{q})$, $q = I(\bar{q})$. Then $\mathscr{V}$ is the direct product of the family $\langle\mathscr{V}_i\rangle_{i\in J}$ with respect to the family $\langle p_i\rangle_{i\in J}$ of homomorphisms.

Proof. It is obvious that p_i is a homomorphism from $\mathscr{V}$ to $\mathscr{V}_i$. Let $\mathscr{V}'$ be a valuation structure and, for every $i \in J$, let $k_i\colon V' \to V_i$ be a homomorphism. We have to prove that there is a homomorphism k such that for every $i \in J$ the following diagram commutes:



Define $k\colon V' \to V$ by setting $k(v) = \langle k_i(v)\rangle_{i\in I}$ for every $v \in V'$. Let $q' \in Q'$ and $X \in D_{q'}$. Then $p_i(k(X)) = k_i(X)$ for every $i \in J$. This proves that $k(X) \in D_q$. Moreover $q(k(X)) = \langle q_i(p_i(k(X)))\rangle_{i\in J} = \langle q_i(k_i(X))\rangle_{i\in J} = \langle k_i(q'(X))\rangle_{i\in J} = k(q'(X))$.

3. Congruences and quotients of valuation structures

A *congruence* ψ of a valuation structure $\mathscr{V} = (V, C, Q)$ is a congruence of the algebraic structure (V, C) compatible with the quantifiers if there are. This means that for every $q \in Q$ and $X, Y \in D_q$

(3.1) $\quad [X]_\psi = [Y]_\psi \Rightarrow [q(X)]_\psi = [q(Y)]_\psi,$

where $[Z]_\psi = \{[z]_\psi \in V/\psi \mid z \in Z\}$ for every $Z \subseteq V$. The *quotient* of $\mathscr{V}$ by ψ is the valuation structure $\mathscr{V}' = \mathscr{V}/\psi = (V', Q', C')$ such that (V', C') is the quotient of (V, C) via ψ and, for every $\bar{q} \in \bar{Q}$, if $q' = I'(\bar{q})$, then $D_{q'} = \{[X]_\psi \mid X \in D_q\}$ and

(3.2) $\quad q'([X]_\psi) = [q(X)]_\psi \quad$ for every $X \in D_q$.

The following proposition shows that the usual homomorphism theorems hold.

Proposition 3.1. *Let $\mathscr{V} = (V, C, Q)$ and let $\mathscr{V}' = (V', C', Q')$ be two valuation structures and $k\colon \mathscr{V} \to \mathscr{V}'$ a homomorphism. Then the relation*

(3.3) $\quad \psi = \{(x, y) \in V^2 \mid k(x) = k(y)\}$

is a congruence relation of $\mathscr{V}$, the kern of k. Conversely, if ψ is a congruence of $\mathscr{V}$ and $\mathscr{V}'$ the relative quotient, the map $k\colon V \to V'$ defined by

(3.4) $\quad k(x) = [x]_\psi \quad$ *for every* $x \in V$

is a homomorphism whose kern is ψ.

The proof is as usual.

4. Fuzzy models

Now we will utilize the valuation structures in order to give a generalized definition of semantics (see [13], [15]). A *(generalized) first order language* is a first order language (in the classical sense) with a type for a valuation structure, i.e. a (generalized) first oder language is given by a system $\langle\langle \bar{F}_m\rangle_{m\in\mathbb{N}}, \langle \bar{R}_m\rangle_{m\in\mathbb{N}}, \bar{Q}, \langle \bar{C}_m\rangle_{m\in\mathbb{N}}\rangle$ of disjoint sets. For any $m \in \mathbb{N}$ the elements of $\bar{F}_m$ and $\bar{R}_m$ will be called *m-argument functors* and *predicates*, respectively. We set $\bar{F} = \bigcup \bar{F}_m$, $\bar{R} = \bigcup \bar{R}_m$. *Terms, open* and *closed formulas* are defined as usual. $\mathscr{L}$ denotes the set of all formulas, $\bar{\mathscr{L}}$ denotes the set off all closed formulas. By $\mathscr{L}_n$ we mean the set of all formulas whose free or bound variables are in $\{x_1, \ldots, x_n\}$ and we write $\alpha[x_1, \ldots, x_n]$ to denote that α belongs to $\mathscr{L}_n$. A *fuzzy model* or a *realization for* $\mathfrak{L}$ is a triple $\mathscr{M} = (D, V, I)$ such that D is a set (the *domain*), V is a set (the *valuation set*) and I (the *interpretation*) is a map such that V together the restriction of I to the type $\bar{Q} \cup (\bigcup \bar{C}_m)$ is a valuation structure, and I associates

a) to every functor $\bar{f} \in \bar{F}_m$ an m-argument operation $f = I(\bar{f})$ of D,
b) to every predicate $\bar{r} \in R_m$ an m-ary V-relation $r = I(\bar{r})$.

We set $F_m = \{I(\bar{f}) \mid \bar{f} \in \bar{F}_m\}$, $R_m = \{I(\bar{r}) \mid \bar{r} \in \bar{R}_m\}$, $F = \bigcup F_m$, $R = \bigcup R_m$. Observe that a fuzzy model M is completely determined by the classical algebra $A(\mathscr{M}) = (D, F)$, the valuation structure $V(\mathscr{M})$ and the set R of fuzzy relations. The valuation of the formulas of $\mathfrak{L}$ with respect to $\mathscr{M} = (D, V, I)$ is defined as follows: If $t(x_1, \ldots, x_n)$ denotes a term whose free variables are in $\{x_1, \ldots, x_n\}$ and $d_1, \ldots, d_n \in D$, then the *value* $t[d_1, \ldots, d_n]$ *of* t *in* $d_1, \ldots, d_n$ *with respect to* $\mathscr{M}$ is defined as usual. If $\alpha \in \mathscr{L}_n$, then $v(\mathscr{M}, \alpha[d_1, \ldots, d_n])$, the *value of* α *in* $d_1, \ldots, d_n$ *with respect to* $\mathscr{M}$ is defined by

(i) $v(\mathcal{M}, \bar{r}(t_1, \ldots, t_p)[d_1, \ldots, d_n]) = r(t_1[d_1, \ldots, d_n], \ldots, t_p[d_1, \ldots, d_n])$,

(ii) $v(\mathcal{M}, \bar{c}(\alpha_1, \ldots, \alpha_s)[d_1, \ldots, d_n]) = c(v(\mathcal{M}, \alpha_1[d_1, \ldots, d_n]), \ldots, v(\mathcal{M}, \alpha_s[d_1, \ldots, d_n]))$,

(iii) $v(\mathcal{M}, \bar{q}x_h\beta[d_1, \ldots, d_n]) = q(\{v(\mathcal{M}, \beta[d_1, \ldots, d_{h-1}, d, d_{h+1}, \ldots, d_n]) \mid d \in D\})$,

for every $p, s \in \mathbf{N}^*$, $\bar{r} \in \bar{R}_p$, $\bar{c} \in \bar{C}_s$, $t_1, \ldots, t_n$ terms, $\alpha_1, \ldots, \alpha_s$, $\beta \in \mathcal{L}_n$, $\bar{q} \in \bar{Q}$, $h \in \{1, \ldots, n\}$.

Observe that if $V(\mathcal{M})$ is not complete, then the valuation can be undefined for some formula. If this is not the case the fuzzy model is called *completely valued.*

In the sequel we suppose that $t, f \in C_0$, $\wedge \in C_2$ and that $(V(\mathcal{M}), \wedge, 0, 1)$ is a semilattice with universal bounds 0 and 1, where $0 = I(f)$, $1 = I(t)$ and the interpretation of $\wedge$ is still denoted by $\wedge$. If $\alpha \in \mathcal{L}_n$ and $v(\mathcal{M}, \alpha[d_1, \ldots, d_n]) = 1$ for every $d_1, \ldots, d_n \in D$, then $\mathcal{M}$ is called a *model of* α. M is a *model of a set S of formulas* if it is a model of α for every $\alpha \in S$. Note that if the valuation structure is the two valued boolean algebra $\mathcal{B}_2$, then the above definition gives the classical concept of semantics. If the valuation structures are the Heyting, Łukasiewicz or modal algebras, we obtain the semantics of the corresponding first order logics. In all these cases it is supposed that the quantifiers are interpreted as the sup and inf operator, respectively. But several other interesting definitions of the quantifiers are possible. For example we can interpret the universal quantifier $\forall$ by the map q defined by

$$q(X) = \begin{cases} 1 & \text{if } X = \{1\}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Also we can interpret the existential quantifier $\exists$ by

$$q(X) = \begin{cases} 1 & \text{if } 1 \in X, \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

In such a manner one obtains logics whose closed formulas are classically valued even if non classical models are also admissible. Another interesting class of quantifiers is defined in Section 5. The expression "fuzzy model" is better than "realization" given in [13] because many objects of investigation of fuzzy set theory are fuzzy models in the above sense. For example fuzzy algebras, free, pure, very pure, left unitary, right unitary fuzzy semigroups are fuzzy models of suitable formulas. The same holds for the fuzzy graphs and the similarity relations. As an example we examine the case of the fuzzy subalgebra. Recall that a *fuzzy subalgebra* (see [4], [14]) is an L-subset $f\colon A \to L$ of an algebra A, where L is the unitary semilattice with zero, such that

$$f(h(x_1, \ldots, x_n)) \geqq f(x_1) \wedge \ldots \wedge f(x_n) \tag{4.3}$$

for every n-ary operation h of A and $x_1, \ldots, x_n \in A$. One proves that an L-subset of A is a fuzzy subalgebra if and only if every cut is a subalgebra. Then to every fuzzy subalgebra we can associate the family of subalgebras $(C(u))_{u \in L}$. Observe that every (characteristic function of a) subalgebra of A is a crisp fuzzy subalgebra, and conversely.

Proposition 4.1. *Assume that $\mathcal{L}$ is a language such that $\to \in \bar{C}_2$, $\forall \in \bar{Q}$, $\bar{R} = \bar{R}_1 = \{\bar{r}\}$. Moreover suppose that the interpretation of $\to$ is an operation $i(x, y)$ such that $i(x, y) = 1$ implies $x \leq y$ and the interpretation of $\forall$ is a map q such that $q(X) = 1$ implies $X = \{1\}$. Then every fuzzy model of the set of formulas*

$$\forall x_1, \ldots, \forall x_n(\bar{r}(x_1) \wedge \ldots \wedge \bar{r}(x_n) \to \bar{r}(s(x_1, \ldots, x_n))) \tag{4.4}$$

with $\bar{s} \in \bar{F}_n$, is a fuzzy algebra. Conversely every fuzzy algebra is a model of this set of formulas with respect a suitable interpretation.

Proof. The first part of proposition is obvious. For the second observe that every fuzzy algebra $g: A \to L$ becomes a fuzzy model of $\mathscr{L}$ provided that we interpret $\bar{r}$ by g, t, f and $\wedge$ by 1, 0 and the semilattice operation $\wedge$ of L, resp., $\to$ by the operation $i: L \times L \to L$ defined by

$$(4.5) \qquad i(u, v) \coloneqq \begin{cases} 1 & \text{if } u \leqq v, \\ 0 & \text{otherwise,} \end{cases}$$

and $\forall$ by (4.1). Such a fuzzy model is a model of (4.4).

Observe that the classical models of (4.4) are characterized by an algebra A and a subalgebra of A (the interpretation of $\bar{r}$).

5. Quantifiers and entropy

In the fuzzy set theory the *entropy* operators have been introduced in order to give a measure of the "degree of fuzziness" of a fuzzy set (see [3], [5], [9]). Now we can define several types of quantifiers that enable us to give such a measure. For example, suppose that a suitable definition of equivalence and negation is given, that $\bar{E} \in \bar{Q}$ and that $E = I(\bar{E})$ is defined by

$$(5.1) \qquad E(X) = \sup(\{v \leftrightarrow \sim v \mid v \in X\}), \quad X \subseteqq V.$$

Then, if $\alpha[x_1]$ is a formula, we can assume the valuation of the formula $\bar{E}x_1\alpha$ as a measure of the degree of fuzziness or the entropy of the predicate corresponding to α. Obviously in a Boolean model all the predicates have zero entropy. We can also substitute $\bar{E}x_1\alpha$ by the more expressive formula $\exists x_1(\alpha \leftrightarrow \sim\alpha)$. We can obtain other examples of entropy by substituting the "contradiction" $v \leftrightarrow \sim v$ by others contradictions, for example

$$(5.2) \qquad E(X) = \sup(\{v \wedge \sim v \mid v \in X\}), \quad X \subseteqq V.$$

In such a manner different types of entropies are defined. For example if $V(\mathscr{M})$ is the Łukasiewicz algebra $\mathscr{B}_\infty$ and $v(\mathscr{M}, \alpha[d]) = \frac{1}{2}$ for a suitable $d \in D$, then the entropy computed by (5.1) is equal to 1, while the entropy computed by (5.2) is equal to $\frac{1}{2}$.

6. The category of fuzzy models

In order to organize the class of the fuzzy models of a given language $\mathscr{L}$ in a category $F(\mathscr{L})$, we have to define the morphism concept. A *morphism* from a fuzzy model $\mathscr{M} = (D, I)$ into a fuzzy model $\mathscr{M}' = (D', I')$ is a pair (h, k) of homomorphisms from $A(\mathscr{M})$ into $A(\mathscr{M}')$ and from $V(\mathscr{M})$ into $V(\mathscr{M}')$ respectively, such that the following diagram commutes

$$(6.1) \qquad \begin{array}{ccc} D^n & \xrightarrow{h} & D'^n \\ r\downarrow & & \downarrow r' \\ V & \xrightarrow{k} & V' \end{array}$$

for every $\bar{r} \in \bar{R}_n$, where $r = I(\bar{r})$, $r' = I'(\bar{r})$ and $h(d_1, \ldots, d_n)$ is $(h(d_1), \ldots, h(d_n))$ for every $(d_1, \ldots, d_n) \in D^n$. The *product* of two morphisms (h, k) and (h', k') is defined by $(h, k) \cdot (h', k') = (hh', kk')$.

If $\bar{R} = \bar{R}_1 = \{\bar{r}\}$, $\forall$ is in $\bar{Q}$ and $\to$ is in $\bar{C}$, then an interesting subcategory of $F(\mathscr{L})$ is the *category of the fuzzy algebras of type* $\mathscr{L}$, i.e. the full subcategory of $F(\mathscr{L})$ whose objects are the models of (4.4), where $\forall$ and $\to$ are interpreted by (4.1) and (4.5), respectively.

A morphism (h, k) such that h and k are both monomorphisms, epimorphisms or isomorphisms is called *monomorphism*, *epimorphism* or *isomorphism*, respectively. A morphism is *elementary* if

$$v(\mathscr{M}', \alpha[h(d_1), \ldots, h(d_n)]) = k(v(\mathscr{M}, \alpha[d_1, \ldots, d_n])) \tag{6.2}$$

for every formula $\alpha \in \mathscr{L}_n$ and $d_1, \ldots, d_n \in D$. If (h, k) is an elementary monomorphism, $\mathscr{M}'$ is called an *elementary extension* of $\mathscr{M}$. If h and k are the identity embedding, $\mathscr{M}$ is called an *elementary submodel* of $\mathscr{M}'$.

Now, we give a condition in order that a morphism is elementary.

Proposition 6.1. *Let $\mathscr{M}$ and $\mathscr{M}'$ be completely valued fuzzy models and (h, k) a morphism from $\mathscr{M}$ to $\mathscr{M}'$. Moreover, assume that, for every $n \in \mathbb{N}^*$, $\alpha \in \mathscr{L}_n$, $i \in \{1, \ldots, n\}$, $\bar{q} \in \bar{Q}$ and $d_1, \ldots, d_n \in D$,*

$$\begin{aligned} v(\mathscr{M}', \bar{q}x_i\beta[h(d_1), \ldots, h(d_n)]) \\ = q'\{v(\mathscr{M}', \beta[h(d_1), \ldots, h(d_{i-1}), h(d), h(d_{i+1}), \ldots, h(d_n)]) \mid d \in D\}. \end{aligned} \tag{6.3}$$

Then (h, k) is elementary.

Proof. We prove (6.2) by induction on the complexity of α. Suppose that α is atomic, i.e. of type $\bar{r}(t_1, \ldots, t_p)$ with $\bar{r} \in \bar{R}_p$ and $t_1, \ldots, t_p$ terms. Since h is a homomorphism, we have for every $d_1, \ldots, d_n \in D$ and $i = 1, \ldots, p$ that $t'_i(h(d_1), \ldots, h(d_n)) = h(t_i(d_1, \ldots, d_n))$, with obvious meaning of the symbols. Then, by the commutativity of (4.1),

$$\begin{aligned} v(\mathscr{M}', \bar{r}(t_1, \ldots, t_p)\,[h(d_1), \ldots, h(d_n)]) &= r'(t'_1[h(d_1), \ldots, h(d_n)], \ldots, t'_p[h(d_1), \ldots, h(d_n)]) \\ &= r'(h(t_1[d_1, \ldots, d_n]), \ldots, h(t_p[d_1, \ldots, d_n])) \\ &= k(r(t_1[d_1, \ldots, d_n], \ldots, t_p[d_1, \ldots, d_n])) \\ &= k(v(\mathscr{M}, \bar{r}(t_1, \ldots, t_p)\,[d_1, \ldots, d_n])). \end{aligned}$$

Suppose that $\alpha = \bar{c}(\alpha_1, \ldots, \alpha_p)$, then

$$\begin{aligned} v(\mathscr{M}', \alpha[h(d_1), \ldots, h(d_n)]) &= c'(v(\mathscr{M}', \alpha_1[h(d_1), \ldots, h(d_n)]), \ldots, v(\mathscr{M}', \alpha_p[h(d_1), \ldots, h(d_n)])) \\ &= c'(k(v(\mathscr{M}, \alpha_1[d_1, \ldots, d_n])), \ldots, k(v(\mathscr{M}, \alpha_p[d_1, \ldots, d_n]))) \\ &= k[c(v(\mathscr{M}, \alpha_1[d_1, \ldots, d_n]), \ldots, v(\mathscr{M}, \alpha_p[d_1, \ldots, d_n]))] \\ &= k[v(\mathscr{M}, \alpha[d_1, \ldots, d_n])]. \end{aligned}$$

The proof for the quantifiers $\bar{q} \in \bar{Q}$ is analogous.

Proposition 6.2. *Let $\mathscr{M}$ be completely valued and (h, k) a morphism from $\mathscr{M}$ to $\mathscr{M}'$ with h surjective. Then $\mathscr{M}'$ is completely valued and (h, k) is elementary.*

Proof. It suffices to repeat the proof of Proposition 6.1. The unique difference is that in this case we have additionally to prove that $\mathscr{M}'$ is completely valued.

7. Direct products of fuzzy models

Now we will prove that the category $F(\mathscr{L})$ of the fuzzy models of $\mathscr{L}$ has direct products.

Proposition 7.1. *The category $F(\mathscr{L})$ has direct products. Namely, the direct product of a family $\langle \mathscr{M}_i \rangle_{i\in J}$ of fuzzy models is the fuzzy model $\mathscr{M}$ such that $A(\mathscr{M})$ and $V(\mathscr{M})$ are the direct products of $\langle A(\mathscr{M}_i)\rangle_{i\in J}$ and $\langle V(\mathscr{M}_i)\rangle_{i\in J}$, respectively, and, for every $\bar{r} \in \bar{R}_n$, $r = I(\bar{r})$ is defined by $r(d_1, \ldots, d_n) = \langle r_i(d_1^i \ldots, d_n^i)\rangle_{i\in J}$ for every $d_1 = \langle d_1^i\rangle_{i\in J}, \ldots, d_n = \langle d_n^i\rangle_{i\in J} \in \mathscr{M}$, where $r_i = I_i(\bar{r})$.*

Proof. Let $p_i\colon D \to D_i$ and $p_i'\colon V \to V_i$ be the i-projections. To prove that (p_i, p_i') is a morphism, it suffices to observe that

$$p_j'(r(d_1, \ldots, d_n)) = p_j'(\langle r_i(d_1^i, \ldots, d_n^i)\rangle_{i\in J}) = r_j(d_1^j, \ldots, d_n^j) = r_j(p_j(d_1), \ldots, p_j(d_n)).$$

Let $\mathscr{M}' = (D', V', I')$ be any fuzzy model and for every $i \in J$ let (h_i, k_i) be a morphism from $\mathscr{M}'$ to $\mathscr{M}_i$. Then it is immediate that the pair (h, k) defined by $h(x) = \langle h_i(x)\rangle_{i\in J}$ and $k(v) = \langle k_i(v)\rangle_{i\in J}$ for every $x \in D'$ and $v \in V'$, is the desired morphism from $\mathscr{M}'$ to $\mathscr{M}$.

The following proposition is proved in [16].

Proposition 7.2. *Let $\langle \mathscr{M}_i\rangle_{i\in J}$ be a completely valued family of fuzzy models, then the direct product $\mathscr{M}'$ is completely valued and*

$$(7.2)\qquad v(\mathscr{M}', \alpha[d_1, \ldots, d_n]) = \langle v(\mathscr{M}_i, \alpha[d_1^i, \ldots, d_n^i])\rangle_{i\in J}$$

for every formula $\alpha \in \mathscr{L}_n$ and every $d_1 = \langle d_i^1\rangle_{i\in J}, \ldots, d_n = \langle d_i^n\rangle_{i\in J} \in D$.

The following proposition shows that also the category of the fuzzy subalgebras has direct products.

Proposition 7.3. *The direct product of a family of fuzzy algebras is a fuzzy algebra, i.e. the category of fuzzy algebras has direct products.*

Proof. It follows from Proposition 4.1 and Proposition 7.2.

Proposition 7.4. *Let $\mathscr{M}$ be a completely valued fuzzy model, J a set and $\mathscr{M}'$ the direct power of $\mathscr{M}$ with index set J. Then, for every $d_1, \ldots, d_n \in D$ and $\alpha \in \mathscr{L}_n$,*

$$(7.3)\qquad v(\mathscr{M}', \alpha[d_1, \ldots, d_n]) = v(\mathscr{M}, \alpha[d_1, \ldots, d_n]),$$

where we have identified the elements of $\mathscr{M}$ and V with the relative constant maps. In other words, $\mathscr{M}'$ is an elementary extension of $\mathscr{M}$.

Proof. Obvious.

8. Congruences and quotients of fuzzy models

A *congruence of a fuzzy model $\mathscr{M}$* is a pair (θ, ψ) of congruences of $A(\mathscr{M})$ and $V(\mathscr{M})$, respectively, such that for every $r \in R_n$ and $d_1, \ldots, d_n, d_1', \ldots, d_n' \in D$

$$(8.1)\qquad d_1 \equiv_\theta d_1', \ldots, d_n \equiv_\theta d_n' \Rightarrow r(d_1, \ldots, d_n) \equiv_\psi r(d_1', \ldots, d_n').$$

The *quotient $\mathscr{M}/(\theta, \psi)$ of $\mathscr{M}$ by (θ, ψ)* is the fuzzy model $\mathscr{M}'$ such that $A(\mathscr{M}')$ and $V(\mathscr{M}')$ are the quotients of $A(\mathscr{M})$ and $V(\mathscr{M})$ by θ and ψ, respectively, and

$$r'([d_1]_\theta, \ldots, [d_n]_\theta) = [r(d_1, \ldots, d_n)]_\psi$$

for every $\bar{r} \in \bar{R}$, $d_1, \ldots, d_n \in D$. Observe that the above defined congruences are not a generalization of the usual ones. As in the classical case, we can prove the homomorphism theorems.

Proposition 8.1. *Let (h, k) be a morphism from $\mathcal{M}$ to $\mathcal{M}'$, then the pair (θ, ψ), where*

$$(8.2) \qquad \theta = \{(d, d') \in D^2 \mid h(d) = h(d')\}, \qquad \psi = \{(v, v') \in V^2 \mid k(v) = k(v')\},$$

is a congruence of $\mathcal{M}$, the kern of (h, k). Conversely, if (θ, ψ) is a congruence of $\mathcal{M}$ and $\mathcal{M}'$ is the relative quotient, then the pair (h, k) defined by $h(d) = [d]_\theta$, $k(v) = [v]_\psi$ for every $d \in D$ and $v \in V$, is an epimorphism whose kern is (θ, ψ).

Proof. It is obvious that θ and ψ are congruences of $A(\mathcal{M})$ and $V(\mathcal{M})$, respectively. Let $\bar{r} \in \bar{R}_n$, $d_1, \ldots, d_n$, $d'_1, \ldots, d'_n \in D$ and assume that $d_1 \equiv_\theta d'_1, \ldots, d_n \equiv_\theta d'_n$. Then $h(d_1) = h(d'_1), \ldots, h(d_n) = h(d'_n)$ and therefore, by the commutativity,

$$k(r(d_1, \ldots, d_n)) = r'(h(d_1), \ldots, h(d_n)) = r'(h(d'_1), \ldots, h(d'_n)) = k(r(d'_1, \ldots, d'_n)),$$

where $r = I(\bar{r})$ and $r' = I'(\bar{r})$. It follows that $r(d_1, \ldots, d_n) \equiv_\psi r(d'_1, \ldots, d'_n)$. This proves that (θ, ψ) is a congruence. Let (θ, ψ) be a congruence, then it is immediate that the maps h and k, defined by (8.3) are homomorphisms. Then, since

$$k(r(d_1, \ldots, d_n)) = [r(d_1, \ldots, d_n)]_\psi = r'([d_1]_\theta, \ldots, [d_n]_\theta) = r'(k(d_1), \ldots, k(d_n)),$$

the pair (h, k) is a morphism from $\mathcal{M}$ to $\mathcal{M}'$. It is obvious that the kern of (h, k) is (θ, ψ).

Proposition 8.2. *Let $\mathcal{M}$ be completely valued and (θ, ψ) a congruence of $\mathcal{M}$. Then the relative quotient $\mathcal{M}'$ is completely valued and*

$$v(\mathcal{M}', \alpha[[d_1]_\theta, \ldots, [d_n]_\theta]) = [v(\mathcal{M}, \alpha[d_1, \ldots, d_n])]_\psi$$

for every formula $\alpha \in \mathcal{L}_n$ and $d_1, \ldots, d_n \in D$.

Proof. It follows from Proposition 6.2 and Proposition 8.1.

Proposition 8.3. *Every quotient of fuzzy subalgebra is a fuzzy subalgebra.*

Proof. It follows from Proposition 8.2 and Proposition 4.1.

Proposition 8.3 shows that in the categories of the fuzzy subalgebras of a given type τ suitable definitions of congruence and quotient exist.

9. Sharpened and crisp versions of a fuzzy model

If θ is the equality relation in D, (θ, ψ) is a congruence of $\mathcal{M}$ iff ψ is a congruence of $V(\mathcal{M})$. We write ψ to denote (θ, ψ) and the relative quotient $\mathcal{M}'$ is called a *sharpened version* of $\mathcal{M}$. If, in particular, $V(M')$ is the two elements Boolean algebra $\mathcal{B}_2$, then M' is a classical model and it is called a *crisp version* of $\mathcal{M}$. Then we obtain the crisp versions of $\mathcal{M}$ by suitable homomorphisms $k\colon V(\mathcal{M}) \to \mathcal{B}_2$ or, equivalently, by suitable congruences of $V(\mathcal{M})$ with only two equivalence classes T, F.

Roughly speeking, to obtain a crisp version it suffices to identify a suitable set T of truth values with 1 and its complement F with 0.

Obviously, in order to find a crisp version of M it is necessary that $V(M)$ have the same type of $\mathcal{B}_2$. This is not very restrictive, indeed we can consider $\mathcal{B}_2$ as a valuation structure of a very large quantity of types. For example, if $V(\mathcal{M})$ is a semilattice we can consider $\mathcal{B}_2$ as a semilattice. If the type of $V(\mathcal{M})$ has two different implications, then we can interpret both these implications by the operation of implication in $\mathcal{B}_2$, and so on.

The following proposition is a particular case of Proposition 8.2.

Proposition 9.1. *If $\mathcal{M}$ is a completely valued fuzzy model and $\mathcal{M}'$ a sharpened version of $\mathcal{M}$ via ψ, then $\mathcal{M}'$ is completely valued and for every $\alpha \in \mathcal{L}_n$ and $d_1, \ldots, d_n \in D$*

$$(9.1) \qquad v(\mathcal{M}', \alpha[d_1, \ldots, d_n]) = [v(\mathcal{M}, \alpha[d_1, \ldots, d_n])]_\psi .$$

Thus in a sense the sharpened and crisp versions of a fuzzy model have the same first order properties of the fuzzy model. In particular it is interesting to observe that every crisp version of a fuzzy algebra $f: A \to L$ is a classical model of (4.4), i.e. a subalgebra of A.

Obviously it is possible that $\mathcal{M}$ has no sharpened or scrisp versions. Examples are given by the following trivial proposition.

Proposition 9.2. *If there exists an algebraic equation which is valid in $V(\mathcal{M})$ and fails in $\mathcal{B}_2$, then there exists no crisp version of $\mathcal{M}$.*

Proof. It sufficies to observe that the congruences preserve the algebraic equations.

As an example, if in $v(\mathcal{M})$ there exists v such that $v = \sim v$, then there is no crisp version of $\mathcal{M}$. This happens, for example, in the Łukasiewicz algebra $\mathcal{B}_\infty$. Indeed $\sim\frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}$. Now, there are various manners in order to avoid such inconveniences.

The first is to consider $\mathcal{M}$, if this is possible, as a fuzzy model valued in a suitable substructure $\mathcal{V}'$ of $V(\mathcal{M})$ for which there exists the desired congruence. For example, if $\mathcal{M}$ is a fuzzy model such that $V(\mathcal{M}) = \mathcal{B}_\infty$ but there exists a $\delta > 0$ such that all the fuzzy predicates possess their values in $V' = [0, \frac{1}{2} - \delta] \cup [\frac{1}{2} + \delta, 1]$, then it is immediate that we can consider $\mathcal{M}$ as valued in V'. Moreover the map $k: V' \to \{0, 1\}$ defined by setting $k(x) = 0$ if $x \leqq \frac{1}{2} - \delta$ and $k(x) = 1$ otherwise is an epimorphism preserving the usual lattice operations and quantifiers. To this epimorphism corresponds a classical model that is reasonable to consider as a crisp version of $\mathcal{M}$.

The second manner is to consider $\mathcal{M}$ as a fuzzy model valued in a valuation structure with a more little type, for example avoid of negation. In other words, we can require that the congruence preserves only a part of the connectives and the quantifiers. In this case the crisp versions of M verify only the first order properties of M expressed in the corresponding sublanguage of $\mathcal{L}$.

Finally we can proceeds as follows: We consider any map $k: V \to \{0, 1\}$ and successively we give a new interpretation of those connectives and quantifiers that are not preserved by k. In such a manner a new valuation structure $\mathcal{V}'$ is defined in V in such a way that $k: \mathcal{V}' \to \mathcal{B}_2$ is an epimorphism. Since it is possible to consider $\mathcal{M}$ as a fuzzy model valued in $\mathcal{V}'$, we consider the classical model associated to k as a crisp version of $\mathcal{M}$. If in $V(\mathcal{M})$ we have a semilattice operation $\wedge$, then it is not too restrictive to suppose that, for every $u, v \in V$, $k(u \wedge v) = k(u) \wedge k(v)$ and $k(0) = 0$, $k(1) = 1$. Indeed this is equivalent to suppose that k is the characteristic function of a proper filter $\mathcal{F}$. If the interpretation of $\to$, $\sim$, $\forall$ are not compatible with k, then we can give new interpretations by setting, for example,

$$(9.2) \qquad c(u, v) = \begin{cases} 0 & \text{if } u \in \mathcal{F} \text{ and } v \notin \mathcal{F}, \\ 1 & \text{otherwise}, \end{cases} \qquad (9.3) \qquad n(u) = \begin{cases} 0 & \text{if } u \in \mathcal{F}, \\ 1 & \text{if } u \notin \mathcal{F}, \end{cases}$$

$$(9.4) \qquad q(X) = \begin{cases} 1 & \text{if } X \subseteqq \mathcal{F}, \\ 0 & \text{otherwise} \end{cases}$$

for every $u, v \in V$ and $X \subseteq V$. In other words if $V(\mathcal{M})$ is a semilattice and $\mathcal{F}$ a filter of $V(\mathcal{M})$, then, by new suitable interpretations of the connectives and quantifiers, we can define a crisp version of $\mathcal{M}$, the *crisp version associated to the filter* $\mathcal{F}$.

To give an example of application of such a type of technique, let $\mathcal{M}$ be a fuzzy algebra and $\mathcal{F}$ a filter of $V(\mathcal{M})$. Then it is immediate that $\mathcal{M}$ is a model of the formulas (4.4) with respect the interpretations (9.2) and (9.4). Then the crisp version $\mathcal{M}'$ of $\mathcal{M}$ associated to $\mathcal{F}$ is a model of (4.4) and therefore a crisp fuzzy algebra. This means that we can identify $\mathcal{M}'$ with a suitable subalgebra of $A(\mathcal{M})$, namely the subalgebra $\{x \in A(\mathcal{M}) \mid r(x) \in \mathcal{F}\}$, where $r = I(\bar{r})$. In particular, if $u \in V(\mathcal{M})$ and $\mathcal{F} = \{v \in V(\mathcal{M}) \mid v \geqq u\}$ is the principal filter generated by u, then we can identify the correspondent crisp version with the cut $\{x \in A(\mathcal{M}) \mid r(x) \geqq u\}$. This suggests the name of *generalized cut* for the subset $\{x \in A(\mathcal{M}) \mid r(x) \in \mathcal{F}\}$.

In conclusion, we have proved the following trivial proposition that generalizes a known property of fuzzy algebras.

Proposition 9.3. *If $\mathcal{M}$ is a fuzzy algebra, then every crisp version associated to a filter is a fuzzy algebra. Equivalentely, every generalized cut of a fuzzy algebra is a subalgebra.*

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